

Black holes vs. naked singularities formation in collapsing Einstein's clusters

S. Jhingan*and G. Magli†

*Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci, 32, 20133 Milano, Italy*

February 7, 2008

Abstract

Non-static, spherically symmetric clusters of counter-rotating particles, of the type first introduced by Einstein, are analysed here. The initial data space can be parameterized in terms of three arbitrary functions, namely; initial density, velocity and angular momentum profiles. The final state of collapse, black hole or naked singularity, turns out to depend on the order of the first non-vanishing derivatives of such functions at the centre. The work extends recent results by Harada, Iguchi and Nakao.

*E-mail : sanjhi@mate.polimi.it

†E-mail : magli@mate.polimi.it

1 Introduction

The “final fate” of gravitational collapse of spherically symmetric dust clouds is well understood. The central singularity can be naked or covered depending on the choice of initial data; there is a “critical branch” of solutions where a transition from naked singularities to black holes occurs (see [1, 2] and references therein).

Recently, a considerable effort has been paid in order to extend the above mentioned results to collapse models with general matter fields. In particular, one would like to understand the role played by stresses in the formation and the nature of the singularities, since both numerical [3, 4, 5] and analytical [6, 7] studies tend to confirm violations of the cosmic censorship conjecture of the kind observed in dust spacetimes.

Gravitational collapse with stresses is quite difficult to handle analytically even in spherical symmetry, one of the reasons being the lack of physically valid exact solutions. In a recent series of papers [8, 9] we have obtained the general exact solution for the case of anisotropic materials sustained only by tangential stresses (see [10] for a review on the role of anisotropic materials in relativistic astrophysics); the nature of the singularities forming in such solutions is still largely unknown (exceptions are the self-similar case [9] and models with special equation of state [11, 12]).

Among the solutions with vanishing radial stresses, there is a system of counter-rotating particles - the “Einstein cluster” - first introduced by Einstein [13] (see also [14]) in the static case and then generalized to the non-static case by Datta [15] and Bondi [16] (see also [17] and [18]). The motion of the particles in the cluster is sustained by angular momentum whose average effect is to introduce a non-vanishing tangential pressure in the energy-momentum tensor. This model is particularly interesting from the physical point of view since it can give some hint on the effects of rotation on collapse, without raising difficulties usually associated with the axisymmetric spacetimes. The analysis of the singularities of the Einstein cluster model has been recently pioneered by Harada, Iguchi and Nakao ([19], to be referred afterwards as HIN). Using the “mass-area” coordinate technique, developed in [9], these authors discovered a particular solution within the Einstein cluster class for which the quadratures can be explicitly carried out. This allowed them to discuss the nature of the singularity for this special solution in full details. They have also discovered several features of the singularities forming in the general model, for which the metric is not expressible in terms of elementary functions. Their results comprise several new examples of naked singularities.

Our results here extend the HIN results to non-analytic cases. We show that the initial data can be parameterized in terms of the first non-vanishing derivatives near the centre, as in the dust case, and obtain the corresponding spectrum of endstates of the collapse for marginally bound initial configurations. Several new interesting features of the Einstein cluster model arise.

2 The Einstein cluster model

The general solution of the Einstein field equations for spherically symmetric collapse with tangential stresses can be reduced to quadratures using a technique first introduced by Ori [20] to study the spherically symmetric charged dust. We will very briefly recall the

solution here, referring the reader to paper [9] for details.

Systems with vanishing radial stresses are characterized by conservation of the Misner-Sharp mass (m). Therefore, one can use it as a comoving label and introduce a system of coordinates (“mass-area coordinates”) in which the line element turns out to be

$$ds^2 = -K^2 \left(1 - \frac{2m}{R}\right) dm^2 + 2\frac{KE}{uh} dR dm - \frac{1}{u^2} dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1)$$

Here K , u and h are functions of R and m . The function u is the modulus of the velocity of the collapsing shells and satisfies

$$u^2 = -1 + \frac{2m}{R} + \frac{E^2}{h^2}, \quad (2)$$

while

$$K = g(m) + \int G(m, R) dR, \quad (3)$$

where

$$G(m, R) := \frac{h}{RE} \left[1 + \frac{R}{2} \left(\frac{E^2}{h^2} \right)_{,m} \right] \left(-1 + \frac{2m}{R} + \frac{E^2}{h^2} \right)^{-3/2}. \quad (4)$$

The functions $g(m)$ and $E(m)$ are arbitrary. The quantity $h(m, R)$ is a measure of the internal energy stored in the material, and therefore plays the role of equation of state. It can be shown that the quantities characterizing a material with vanishing radial stresses, namely the energy density ϵ and the tangential stress Π , can be written in terms of h as follows [8]:

$$\epsilon = \frac{h}{4\pi u K E R^2}, \quad \Pi = -\frac{R}{2h} \frac{\partial h}{\partial R} \epsilon. \quad (5)$$

These formulae show that the tangential stress vanishes whenever the function h does not depend on R . Therefore in this case the material is not sustained by any stress and as a consequence the line element (1) reduces to the dust one (Lemaître-Tolman-Bondi spacetime). Since, in all formulae, E and h appear only in ratio, the value of h on an arbitrary curve $R = R_0(m)$ can be rescaled to unity, so that, in particular, the dust solutions can be characterized by $h = 1$. The function $R_0(m)$ plays the same role as played by the initial mass distribution in the “standard” comoving coordinates (the inverse transformation from mass label to lagrangian label being $r = R_0(m)$) while the function $E(m)$ plays the role of “energy function” familiar from dust models (see [9] for details).

The above recalled structure shows that a solution with tangential stresses is identified (obviously modulo gauge transformations) by a triplet of functions $\{g, E, h\}$. This “parameterization” contains the dust spacetimes as the “subspace” $\{g, E, 1\}$. In this way we can construct in a mathematically precise way, a comparison between dust and tangential stress evolutions by comparing the end state of the dust collapse ($\{g, E, 1\}$ with chosen g and E) with the endstates of the tangential stress solutions $\{g, E, h\}$ with the same g and E and different choices of the equation of state h (see [9, 21] for details).

The physical singularities of the spacetime described by the metric (1) correspond to infinite energy density and are given by $R = 0$ or $K = 0$. At $R = 0$ the shells of matter “crush to zero size” (shell-focusing singularities) while vanishing of K implies intersection of different shells of matter (shell-crossing singularities). We are interested here only in the central shell-focusing singularity. It is important to distinguish the central singularity (i.e. that shell focusing singularity occurring at $R = m = 0$) from the non-central singularities (singularities occurring at $R = 0, m \neq 0$) which are always covered in spacetimes with vanishing radial stresses, as a simple consequence of mass conservation.

At $R = m = 0$ the apparent horizon and the singularity form simultaneously. Therefore, to analyse the causal nature of the singularity, one has to study outgoing radial null geodesics and check if there are some which meet the singularity in the past. This can be done [22] by defining a quantity $x = R/2m^\alpha$ (where $1/3 \leq \alpha \leq 1$) and searching for a real positive solution x_0 of the following “root equation”

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{m^{\frac{3}{2}(1-\alpha)}}{2\alpha} \left[-\frac{R_{0,m}h(m, R_0)}{E(m)u(m, R_0)} + \int_{R_0}^{2m^\alpha x} GdR \right] \left(\sqrt{(-1 + \frac{E^2}{h^2})m^{\alpha-1} + \frac{1}{x}} \right) \times \\ \times \left(\frac{E}{h} - \sqrt{-1 + \frac{E^2}{h^2} + \frac{m^{1-\alpha}}{x}} \right). \quad (6)$$

This equation depends both on the choice of the equation of state h and on the initial data. Setting $h = 1$ one can deduce from it the final fate of the dust gravitational collapse [2, 23], so that it can be used to study the way in which the end state of collapse for fixed initial data is altered by a non-trivial h .

To analyse the roots equation, one faces with the fact that the integral appearing in formulae (6) introduces a sort of non-locality which is difficult to handle. As a consequence, it is very difficult to work in full generality and only a few cases have been analysed so far. Among different cases with tangential stresses, a special role is played by a system of counter-rotating particles (The Einstein Cluster). This system has several attractive features from the physical point of view since it mimics in a simple way the effect of rotation (see e.g. [24]). So far, the only known results on the nature of the singularities forming in this model have been obtained in the HIN paper (such results will be recalled in the next Section).

It has been shown in paper [8] that the Einstein cluster model can be obtained from the general exact solution with tangential stresses by choosing a specific form of h :

$$h(m, R) = h_0(m) \left[1 + \frac{L^2(m)}{R^2} \right]^{1/2}. \quad (7)$$

In the above formula, the functions h_0 and L are arbitrary. The function L is the specific angular momentum of the particles while only one among the functions E and h_0 is independent. Therefore the function h_0 can be used to normalize the value of h to unity on the initial data surface $R = R_0(m)$. This choice does not affect the generality of the equation of state (7) within the Einstein cluster class until the function E is chosen.

The final fate of the collapse turns out to depend on the behaviour of the physical quantities near origin, on the initial data surface. Three such quantities are independent, and we shall use from now on F , L^2 and $f = E^2 - 1$. We assume these functions to be

Taylor-expandable with respect to r (the “standard” radial coordinate $r = R_0$) near $r = 0$. It is now easy to check (see [8]) that regularity at the initial surface requires the following kind of expansions:

$$F(r) = F_0 r^3 + F_n r^{n+3} + \dots \quad (8)$$

$$f(r) = f_l r^l + \dots \quad (9)$$

$$L^2(r) = L_k r^k + \dots \quad (10)$$

where $l \geq 2$, $k \geq 4$ and dots denote higher orders terms. The n -term in $F(r)$ has been put in evidence because it characterizes the first non-vanishing derivative of the initial density profile; a positive F_0 and a negative F_n are required for physical reasonability.

In the above formulae we allow for terms of *any parity*. It is very important to specify carefully the meaning of this choice, since a considerable debate on this point has already taken place in the case of dust (i.e. $L^2 = 0$ here).

In the dust case, pioneering numerical results by Eardley and Smarr [25] were first investigated analytically by Christodoulou [26] and Newman [27]. Both these authors required the density and metric functions to be smooth (C^∞) with respect to a system of *cartesian* coordinates near the centre (a function of this kind is sometimes called extended analytic). Obviously, this implies that a power series expansion near $r = 0$ can contain only *even* powers (for instance, if a linear term is absent but a cubic term present in the mass $F(r)$, the initial density $\epsilon(r, 0) = F'/4\pi r^2$ is only C^2 near the centre w.r. to a local cartesian system). From the physical point of view, however, this restriction is unnecessary and actually rules out many interesting situations [28]. Indeed, the results by Christodoulou and Newman were extended to the case of an arbitrary parity by Singh and Joshi [23] showing analytically, for instance, the existence of a transition parameter at $n = 3$ for marginally bound data (this structure will be very briefly recalled in the next section).

Returning to the case of Einstein cluster, the extended analytic case here corresponds to even values of k in the expansion of L^2 and was first considered in the HIN paper, while we allow here for odd values of k as well as for even ones.

The expansions can now be translated in mass–area coordinates. In particular, we put

$$L^2(m) = \beta_k m^{k/3} + \dots, \quad (11)$$

where β_k is a positive constant.

3 The endstate of collapse

The dust models admit no globally regular solutions. This is not true for general matter fields even within spherical symmetry. For example, in homogeneous perfect fluid models, we can have globally regular solutions which satisfy suitable energy conditions and have regular initial data (see e.g. [29]); in other words, the endstate of a non-static model can be an eternally oscillating solution or a bounce-back scenario. It is, therefore, important to understand the qualitative behaviour of the motion as a preliminary step before investigating the causal structure.

The equation of motion (2) governing the collapsing shells can be conveniently written as

$$u^2 = \frac{Z(R, R_0)}{R(R^2 + L^2)}, \quad (12)$$

where the “effective potential” Z is defined by

$$Z(R, R_0) = fR^3 + 2mR^2 + L^2(2m - R). \quad (13)$$

This function should be interpreted as an analogue of the Newtonian effective potential governing the motion of the *fixed* shell R_0 , so that the allowed regions of the motion correspond to $Z \geq 0$. It is immediately seen that the sign of $f = E^2 h_0^2 - 1$ governs the behaviour at large values of R , since unboundedly large values of R are permitted only for $f \geq 0$. The region near $R = 0$ is, instead, always allowed for any $R_0 \neq 0$ since $Z(0, R_0) = 2mL^2 > 0$. To investigate on the formation of the central singularity (i.e. the singularity forming at $R = R_0 = 0$), let $R = \zeta(t, R_0)R_0$. To the lowest order in R_0 one has

$$Z \approx (-\beta_k \zeta R_0^{k-4} + 2F_0 \zeta^2 + f_0 \zeta^3 + 2\beta_k F_0 R_0^{k-2} + \beta_k \zeta^3 R_0^{k-4} + \beta_k f_0 \zeta^3 R_0^{k-2}) R_0^5, \quad (14)$$

(recall that $F = F_0 R_0^3$, $f = f_0 R_0^2$ and $L^2 = \beta_k R_0^k$ to the lowest order). Near $\zeta = 0$, the above equation gives $Z \approx -\beta_4 \zeta R_0^5 < 0$ for $k = 4$, and therefore the singularity does not form. Since it is the positivity of β_4 which does not allow singularity formation, tangential stresses regularize the corresponding ($\beta_4 = 0$) dust solutions. For $k > 4$ one has $Z \approx 2F_0 \zeta^2 R_0^5 > 0$ and the singularity always forms (the case $k = 4$ was firstly studied by Evans [18], while the HIN paper contains the analytic case of even k).

For $k > 4$ we have to analyse further the dynamics of the non-central shells, which depends on the existence of roots of the cubic equation $Z = 0$. Due to the above remarks on the behaviour of Z for big and small values of R , we can *a priori* expect the following kinds of situations. If R_1 , R_2 and R_3 are the roots of the cubic and $R_1 < R_2 < R_3$, one has the following sets of allowed regions

$$\begin{aligned} 0 \leq R \leq R_1, \quad R_2 \leq R \leq R_3, \quad f < 0, \\ 0 \leq R \leq R_1, \quad R_2 \leq R, \quad f \geq 0, \end{aligned} \quad (15)$$

where, of course, the forbidden regions between R_2 and R_3 can disappear if either one or both of these two roots vanish or are not positive (thereby giving $0 \leq R \leq R_1$ for $f < 0$ and $R \geq 0$ for $f > 0$, respectively).

The fine details are quite involved and were given for the first time by Bondi [16]. The point of view here, however, is slightly different since we are interested only in the dynamics of the shells near the central one to check whether the central singularity becomes trapped or not. A simple way to do this is to evaluate the effective potential on curves of the type $R = \lambda F$ with $\lambda > 2$. For the region near centre we obtain

$$Z(\lambda F, R_0) \approx (2 - \lambda)\beta_k F_0 R_0^{k+3} + 2\lambda^2 F_0^3 R_0^9. \quad (16)$$

For $k \leq 5$ the first term is the leading one and it is negative. For $k = 6$ it is possible to find values λF such that $Z(\lambda F, R_0)$ is negative if the quadratic equation $(2 - \lambda)\beta_6 F_0 + 2F_0^3 \lambda^2 = 0$

has a positive root. This in turn implies that the quantity $D := \sqrt{\beta_6}/F_0$ must be greater than four (HIN). In both these cases ($k = 5$ and $k = 6$ for $D > 4$) the central singularity is naked. Indeed, λF is not an allowed value since Z is negative at it. But F goes to zero as R_0^3 , and therefore the shells near the centre must have initial data $R = R_0$ greater than $2F$. For sufficiently small R_0 , this will hold true also for λF , so that a region of the kind $R > R_2 > 2F$ containing $R = R_0$ must exist. But the corresponding dynamics can never cross the forbidden region to reach values smaller than $2F$. Therefore, the region near the centre is untrapped *eternally* in these cases: the apparent horizon does not form and therefore the singularity is naked. This phenomenon has been firstly discovered in the HIN paper for the case $k = 6, D > 4$. Interestingly enough, this kind of eternal singularity is completely different from the naked singularities arising in dust spacetimes, which are usually visible only for a finite amount of time before apparent horizon formation. A peculiar characteristic of the singularities here is that they are not accompanied by non-central singularities, since all the shells near the central one remain regular eternally (the region near $R = 0$ is forbidden for them).

For $k = 6, D \leq 4$ as well as for any $k > 6$ the above argument cannot be applied. The special case $k = 6, D = 4, f = 0$ can, however, be treated analytically since the integral appearing in formula (3) can be expressed in terms of arbitrary functions. The singularity in this case turns out to be always naked; again independently from the choice of the initial matter distribution (HIN).

To understand the nature of the singularities in the remaining possible cases, we have to resort to the root equation. The integral which appears in formula (6) cannot be calculated explicitly, however, the existence of a naked singularity can be probed as follows (for simplicity we shall consider the case $E(m) = 1$). We expand the integrand near centre in the powers of m and express the integral as the sum over successive integrated terms of the series. The proof that this is possible makes use of Lebesgue's dominated convergence theorem, as shown in details in the Appendix.

For $k = 6$ equation (A6) gives

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{1}{2\alpha} \left[\frac{nF_n}{9\sqrt{2}} \frac{m^{(2n-9\alpha+3)/6}}{F_0^{(2n+9)/6}} - \frac{2^{3/2}\beta_6}{15F_0^{1/6}} m^{(7-9\alpha)/6} - \frac{2}{3} x^{3/2} + \dots \right] \left(\frac{m^{1-\alpha}}{x} - \frac{1}{\sqrt{x}} \right), \quad (17)$$

where dots stand for terms going as $\mathcal{O}(m^{\frac{n-1}{3}})$. In this equation it is possible to fix uniquely the value of α for each choice of n , in such a way that the resulting algebraic equation in x_0 has a positive root (see Table 1). Therefore the singularity is always naked for $k = 6$, independently from the choice of the initial density distribution.

As recalled in the previous section, the value of the root gives the tangent of the escaping geodesics near the singularity. It is somewhat unclear if a physical content can be given to the actual ("numerical") value of this tangent, although it is very likely that such an interpretation could be found in the near future. In any case, we can get some insight into the effect of the state equation on naked singularity formation looking *qualitatively* to the role of the parameter β_k in the roots. With this aim, the values of the roots for the various possible initial distributions of mass in presence of a non-zero β_6 (counter-rotating particles) can be compared with the corresponding values for dust ($\beta_6 = 0$) (see Table 1).

| | Initial data $F_0 R_0^3 + F_n R_0^{n+3}$ | root $(x_0)^{3/2}$ | | | |
|---|---|---|-------------|-------------------------------------|-------------|
| | | Einstein cluster | singularity | dust | singularity |
| 1 | n=1 | $\frac{-F_1}{2^{5/2} F_0^{11/6}}$ | visible | $\frac{-F_1}{2^{5/2} F_0^{11/6}}$ | visible |
| 2 | n=2 | $\frac{6\beta_6 F_0^2 - 5F_2}{20\sqrt{2} F_0^{13/6}}$ | visible | $\frac{-F_2}{4\sqrt{2} F_0^{13/6}}$ | visible |
| 3 | n=3 | $\frac{3\beta_6}{10\sqrt{2} F_0^{1/6}}$ | visible | roots for $\xi < \xi_c$ | transition |
| 4 | n > 3 | $\frac{3\beta_6}{10\sqrt{2} F_0^{1/6}}$ | visible | - | black hole |

Table 1: The endstate of counter-rotating particles for $k = 6$ vs dust

At $n = 1$ the inhomogeneity dominates over the effect of counter-rotation: both the endstates are naked singularities and the roots coincide. At $n = 2$ a contribution coming from the counter-rotation appears and both the endstates are still naked singularities. At $n = 3$ the root equation for dust spacetimes becomes a quartic which has positive roots only if the a-dimensional quantity $\xi := F_3/(2^{3/2} F_0^{5/2})$ is less than $\xi_c = -(26 + 15\sqrt{3})/2$ [23]. Therefore, ξ_c can be called critical parameter: at $\xi = \xi_c$ a transition occurs and the endstate of collapse turns from a naked singularity to a black hole. In the case of the Einstein cluster instead the root becomes independent from n at $n = 3$ and is always present for $\beta_6 > 0$ making the endstate always a naked singularity. For $n > 3$ dust always forms a black hole while here the endstate is always a naked singularity.

The existence of such “tenacious” naked singularities which are not sensible to changes in the initial mass distribution, first discovered in the HIN paper, raises the question if these singularities are already present in dust models or are peculiar of spacetimes with stresses. Actually it turns out that such special dust spacetimes exist, as we are going to show.

Observe that the energy function of the HIN model is of the form $f(r) = f_4 r^4 + \dots$ (where $f_4 = -2^4 F_0^2$), i.e., the expansion of this function starts from the quartic terms. Now one can ask for the final fate of those *dust* spacetimes which have initial velocity profiles of this kind. To the best of our knowledge, the answer to this question is not present in the literature since all previous studies on dust spacetimes have assumed a non-vanishing second order term. The causal structure can, however, be analysed easily and it turns out that such solutions actually have the same causal structure of HIN: the singularity is always visible [30].

It remains to consider the cases $k \geq 7$. For $k = 7$ formula (A6) gives

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{1}{2\alpha} \left[\frac{n F_n}{9\sqrt{2}} \frac{m^{(2n-9\alpha+3)/6}}{F_0^{(2n+9)/6}} - \frac{2^{1/2} \beta_7}{5 F_0^{1/6}} m^{3(1-\alpha)/2} - \frac{2}{3} x^{3/2} + \dots \right] \left(\frac{m^{1-\alpha}}{x} - \frac{1}{\sqrt{x}} \right), \quad (18)$$

where dots stand for terms going to zero as $\mathcal{O}(m^{\frac{k+n-7}{3}})$. This equation leads to a structure of the endstates which is similar to that occurring in dust (second column of table one) for $n = 1$ and 2. At the transition ($n = 3$), the region of naked singularities formation is characterized by the inequality $\tilde{\xi} < \xi_c$ where $\tilde{\xi} = \xi - 3\beta_7/(5\sqrt{2} F_0^{1/6})$ and ξ is the dust

parameter recalled above. Since β_7 is positive, a sector of the black hole region in dust is uncovered by the effect of counter-rotation; for $n > 3$ it remains uncovered for strong enough rotation parameter ($\beta_7/F_0^{1/6} > -5\sqrt{2}\xi_c/3$).

Finally, for $k > 7$ the causal structure of the singularity is identical to that of marginally bound Lemaître-Tolman-Bondi spacetimes, since all the terms coming from rotation vanish near the singularity (a similar situation has been recently found to occur in a completely different model with tangential stresses [12]).

4 Discussion

In recent years, a big effort has been undertaken to understand the final fate of the gravitational collapse of a dust cloud. Indeed, one can safely assert that this end state is now completely known in dependence of the choices of the initial data.

The dust model is the most simple model which can be conceived in gravitational collapse and in the absence of general proofs (and indeed even of mathematical formulations of) of cosmic censorship theorems, it was meant to be used as a tool to get insights in to more general collapse situations. We are, therefore, in some sense ready to approach the problem with general stresses and indeed some analytical results are beginning to be known. In this respect our full understanding of dust spacetimes should hopefully be used as a starting point.

As a simple generalization of the dust models, we proposed to analyse spacetimes with tangential stresses and constructed a tool for doing that using mass–area coordinates. The first results on the nature of the singularities for such spacetimes were obtained in the HIN paper for the case of the Einstein cluster model. In the present paper we have extended the HIN results to the non-extended analytic case.

The picture which arises is very intriguing. The key role is played by the parameter k measuring the “strength” of angular momentum near the centre ($L^2 \approx \beta_k R_0^k$ as R_0 tends to zero). For $k = 4$ the solution is regular and tangential stresses avoid singularity formation. For $k = 5$ we have central singularity formation, as in dust, but the neighbouring shells are regular; all such shells are untrapped forever allowing the central singularity to be eternally visible. This phenomenon is clearly connected to singularity theorems since all the solutions considered here satisfy the energy conditions and therefore, in the presence of trapping the shells, would have to become singular. For $k = 6$ a sort of transition takes place, since for $D = \sqrt{\beta_6}/F_0 \leq 4$ also non-central shells can become singular, the exact HIN solution playing the role of separation point. For $k = 7$ the remaining effect of counter-rotation becomes very weak and it is effective only at the $n = 3$ transition from black holes to naked singularities, shifting the transition parameter ξ_c and uncovering part of dust black hole region. The naked singularities continue to exist for strong enough rotation parameter β_7 . For $k > 7$ the final fate is the same as in dust models.

From the physical point of view, this picture is due to the non-convexity of the state function which acts as a “source of acceleration” near the centre avoiding the formation of apparent horizon and therefore enforcing nakedness. The structure of the spectrum of endstates thus depends in a peculiar way on the fact that the Einstein cluster model, when interpreted as a material characterized by a macroscopic equation of state, does not admit

a local minimum of the energy density. This is readily seen since the derivative of the state function h with respect to R is strictly negative and proportional to $-L^2$ near the centre, while equations of state which have to be expected for, say, nuclear matter at high densities should admit a status of (local) minimization of internal energy. However, the HIN results, as well as our results here, do raise the interesting question if the effects of rotation might become dominant with respect to those of a non-trivial equation of state at high densities; when the hypothesis of average geodesic motion (which is the basis of the relative simplicity of the counter-rotating particles model) is lost. To approach this question one should turn to the (so far unsolved) difficulties of the non-spherical collapse scenario.

Remaining in the safety zone of spherical symmetry, our results here support the view that formation of naked singularities has to be expected in *generic* situations of spherical collapse [7, 31].

Acknowledgements

Interesting discussions with Pankaj Joshi are gratefully acknowledged.

Appendix

Consider the function

$$I_1(m; \alpha) := \int_{R_0}^{2m^\alpha x} G(m, R) dR \quad (\text{A1})$$

where $G(m, R)$ is defined by equation (4). Due to the Mean value theorem there exists $\chi(m) \in (R_0, 2m^\alpha x)$ such that

$$I_1(m; \alpha) := (2m^\alpha x - R_0)G(m, \chi(m)). \quad (\text{A2})$$

Since $R_0 \sim m^{1/3}$ as m goes to zero, $1/3 \leq \alpha \leq 1$ and x is finite and positive, both $m^{-1/3}\chi(m)$ and $m^\alpha/\chi(m)$ are finite and positive as m goes to zero. Therefore we can evaluate the right hand side of equation (A2) using (4) as follows

$$\begin{aligned} \int_{R_0}^{2m^\alpha x} G(m, R) dR &= m^{-1} \frac{(2m^{(3\alpha-1)/3}x - F_0^{-1/3} + \dots)}{2\sqrt{2}(1 + \beta_k m^{(k-2)/3} F_0^{2/3})^{1/2}} \left(\frac{\chi(m)}{m^{1/3}} \right)^{1/2} \left[1 + \beta_k m^{(k-6\alpha)/3} \left(\frac{m^\alpha}{\chi(m)} \right)^2 \right. \\ &\quad \left. + \frac{\beta_k}{2} m^{(k-4)/3} \left(\frac{\chi(m)}{m^{1/3}} \right) F_0^{2/3} - \frac{\beta_k}{2} m^{(k-3-3\alpha)/3} \left(\frac{m^\alpha}{\chi(m)} \right) + \dots \right]^{-3/2} \\ &\quad \times \left[\left(1 + \beta_k m^{(k-6\alpha)/3} \left(\frac{m^\alpha}{\chi(m)} \right)^2 \right)^2 + \frac{\beta_k}{6} (k-2) m^{(k-4)/3} \left(\frac{\chi(m)}{m^{1/3}} \right) F_0^{2/3} \right. \\ &\quad \left. - \frac{\beta_k^2}{3} m^{(2k-5-3\alpha)/3} \left(\frac{m^\alpha}{\chi(m)} \right) F_0^{2/3} - \frac{k\beta_k}{6} m^{(k-3-3\alpha)/3} \left(\frac{m^\alpha}{\chi(m)} \right) + \dots \right], \end{aligned} \quad (\text{A3})$$

where dots stand for terms of higher order in positive powers of $m^{n/3}$. For $k \geq 6$ only the factor m^{-1} is divergent, all the other terms being finite due to $\alpha \leq 1$. Therefore, the limit

$$I_2(m; \alpha) := \lim_{m \rightarrow 0} \int_{R_0}^{2m^\alpha x} m G(m, R) dR, \quad (\text{A4})$$

is convergent. Hence, using Lebesgue's dominated convergence theorem, we can expand the integrand near the centre ($m = 0$) in the leading powers of m and integrate successive terms. This gives

$$I_2(m; \alpha) = \lim_{m \rightarrow 0} \left[\frac{-1}{3\sqrt{2}F_0^{1/2}} + \frac{F_n m^{n/3}}{6\sqrt{2}F_0^{(2n+9)/6}} + \frac{\beta_k \sqrt{2}(k-4)}{15F_0^{1/6}} m^{(k-4)/3} - \frac{2}{3} m^{(3\alpha-1)/2} x_0^{3/2} + \frac{\beta_k(2k-9)}{6} m^{(2k-9+3\alpha)/2} x_0^{1/2} + \mathcal{O}(m^{(k-4+n)/3}) \right], \quad (\text{A5})$$

Using this equation in (6), we can write the root equation in the form (valid for $k \geq 6$)

$$x_0 = \lim_{\substack{m \rightarrow 0 \\ R \rightarrow 0}} \frac{m^{\frac{3}{2}(1-\alpha)}}{2\alpha} \left[\frac{nF_n m^{(n-3)/3}}{9\sqrt{2}F_0^{(2n+9)/6}} - \frac{\beta_k \sqrt{2}(k-4)}{15F_0^{1/6}} m^{(k-7)/3} - \frac{2}{3} m^{3(\alpha-1)/2} x_0^{3/2} + \mathcal{O}(m^{(k-7+n)/3}) \right] \left(\frac{m^{1-\alpha}}{x} - \frac{1}{\sqrt{x}} \right). \quad (\text{A6})$$

References

- [1] P. S. Joshi, *Global aspects in gravitation and cosmology*, (Clarendon press, Oxford, 1993).
- [2] S. Jhingan and P. S. Joshi, Ann. Isr. Phys. Soc **13**, 357 (1997).
- [3] A. Ori and T. Piran, Phys. Rev. D **42**, 1068 (1990).
- [4] T. Harada, Phys. Rev. **D58**, 104015 (1998).
- [5] G. Rein, A. Rendall and J. Schaeffer Phys. Rev. **D58**, 044007 (1998).
- [6] F. I. Cooperstock, S. Jhingan, P. S. Joshi and T. P. Singh, Class. Quant. Grav. **14**, 2195 (1997).
- [7] P.S. Joshi and I.H. Dwivedi, Class. Quant. Grav. **16**, 41 (1999).
- [8] G. Magli, Class. Quant. Grav. **14**, 1937 (1997).
- [9] G. Magli, Class. Quant. Grav. **15**, 3215 (1998).
- [10] L. Herrera and N. Santos, Phys. Rep. **286**, 2 (1997).
- [11] T. P. Singh and L. Witten, Class. Quan. Grav. **14**, 3489 (1997).
- [12] S. Barve, T. P. Singh and L. Witten, gr-qc/9901080.
- [13] A. Einstein, Ann. Math. **40**, 922 (1939).
- [14] P. S. Florides, Proc. Roy. Soc. Lon. **A 337**, 529 (1974).
- [15] B. K. Datta, Gen. Rel. Grav. **1**, 19 (1970).
- [16] Bondi, H., Gen. Rel. Grav. **2**, 321 (1971).
- [17] L. Herrera and N. Santos, Gen. Rel. Grav. **27** 1071 (1997).
- [18] A. B. Evans, Gen. Rel. Grav. **8**, 155 (1977).
- [19] T. Harada, H. Iguchi and K. Nakao, Phys. Rev. D **58**, R 041502 (1998).
- [20] A. Ori, Class. Quant. Grav. **7**, 985 (1990).
- [21] S. Jhingan and G. Magli, gr-qc/9903103.
- [22] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D. **47**, 5357 (1993).
- [23] T. P. Singh and P. S. Joshi, Class. Quant. Grav. **13**, 559 (1996).
- [24] G. L. Comer and J. Katz, Class. Quant. Grav. **10**, 1751 (1993).
- [25] D. M. Eardley and L. Smarr, Phys. Rev. **D19**, 2239 (1979).
- [26] D. Christodoulou, Comm. Math. Phys. **93**, 171 (1984).
- [27] R. P. A. C. Newman, Gen. Rel. Grav. **3**, 527 (1986).
- [28] S. Jhingan, P. S. Joshi and T. P. Singh, Class. Quant. Grav. **13**, 3057 (1996).

- [29] W. B. Bonnor and M. C. Faulkes, Mon. Not. Roy. Soc. **137**, 239 (1967).
- [30] Using the root equation a simple calculation shows that the values of the roots are the following: $-F_1/(2^{5/2}F_0^{11/6})$ for $n = 1$, $(24F_0^2 - 5F_2)/(20\sqrt{2}F_0^{13/6})$ for $n = 2$ and $6/(5\sqrt{2}F_0^{1/6})$ for any $n \geq 3$.
- [31] I.H. Dwivedi and P. S. Joshi, Comm. Math. Phys. **166**, 117 (1994).